Final Exam: MAT 319

Instructions: Complete all problems below. You may not use calculators or other aides, including cell phones and books. Show all of your work. **Be sure to write your name and student ID on each page that you hand in.**

1.(20pts) Determine if the following limit exists, and if it does exist find its value:

$$\lim_{x\to\infty} \left(e^x + x\right)^{1/x}.$$

Any theorems that are used must be fully justified by verifying all hypotheses.

Consider $f(x) = \ln(e^x+x)^{1/x} = \frac{1}{x}\ln(e^x+x)$ for $x>0$.

Since $\lim_{x\to\infty} \ln(e^x+x) = \lim_{x\to\infty} x = \infty$, the denominator and its derivative do not vanish, and both numerator and denominator are smooth, we may apply L'Hospital's rule. It follows that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(e^x+x)^{-1}(e^x+1)}{1}$$

$$= \lim_{x\to\infty} \frac{1+e^{-x}}{1+xe^{-x}} = 1.$$

Since exp is continuous we have

$$\lim_{x\to\infty} (e^x+x)^{1/x} = e^{\lim_{x\to\infty} \ln(e^x+x)^{1/x}} = e.$$

2.(20pts) Find a Taylor series expansion for $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ and prove that it converges to $\sinh x$ for all $x \in \mathbb{R}$.

Let $f(x) = \sinh x$ then $f'(x) = \cosh x$, $f''(x) = \sinh x$

and clearly $f^{(2n+1)}(x) = \cosh x$, $f^{(2n)}(x) = \sinh x$, $n \in \mathbb{N}$.

Thus the Taylor series at $x_0 = 0$ is

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$ since $\cosh(0) = 1$, $\sinh(0) = 0$.

The mean value form of the remainder is

$$R_n(x) = \frac{f^{(n)}(x_*)}{n!} x^n$$ for some $x_* \in (0, x)$.

Since $f^{(n)}(x_*) = \cosh x_*$ or $\sinh x_*$ we have

$$\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} C \frac{|x|^n}{n!} = 0 \quad , \quad x \in \mathbb{R} .$$

By Taylor's Theorem the series converges to $\sinh x$ for all $x \in \mathbb{R}$.

3.(20pts) Show that if f is integrable on $[a,b]$, then f is integrable on every interval $[c,d] \subset [a,b]$.

Since f is integrable on $[a,b]$, given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that partitions P with $\text{mesh}(P) < \delta$ satisfy $U(f,P) - L(f,P) < \varepsilon$. This is the "Cauchy Criterion" result. Let P' be a partion of $[c,d]$ with $\text{mesh}(P') < \delta$. Then we may extend P' to a partition P of $[a,b]$ with $\text{mesh}(P) < \delta$. Then

$$U(f,P') - L(f,P') \leq U(f,P) - L(f,P) < \varepsilon .$$

Hence by the other direction in the Cauchy Criterion Theorem, we find that f is integrable on $[c,d]$.

4.(20pts) Prove the following generalization of the Intermediate Value Theorem for Integrals. If f and g are continuous functions on $[a, b]$ and $g(x) \geq 0$ for all $x \in [a, b]$, then there exists $c \in (a, b)$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Assume that $g \not\equiv 0$ otherwise it is automatically true.

Let $M = \max\{f(x) \mid x \in [a,b]\}$ and

$m = \min\{f(x) \mid x \in [a,b]\}$.

Then $m \leq \dfrac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M$. Thus

by the mean value theorem $\exists\ c \in (a,b)$ such

that $\dfrac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} = f(c)$.

5.(20pts) Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_{x-1}^{x+1} f(t)dt \quad \text{for all } x \in \mathbb{R}.$$

Prove that F is differentiable on $\mathbb{R}$ and compute F'.

Observe that $F(x) = \int_c^{x+1} f(t)dt - \int_c^{x-1} f(t)dt$

where $c \in (x-1, x+1)$ is fixed. Since f is continuous, the Fundamental Theorem of Calculus implies that the two integrals are differentiable (hence F is differentiable) and

$$F'(x) = f(x+1) - f(x-1).$$